Lecture 4 – Accretion

Accretion is the process where a massive object continuously accumulates more mass by gravitationally attracting nearby gas. Accretion onto compact objects (e.g., black holes, neutron stars, and white dwarfs) is the most efficient source of energy in the universe. Consider a compact object of mass M with a radius r_* accreting a gas cloud of mass m onto its surface, the energy release is the gravitational potential energy GMm/r_*. If we express it in terms of the rest mass energy of the cloud (mc^2 where c is the speed of light), the available energy is $0.5(r_s/r_*)mc^2$ where $r_s = 2GM/c^2 = 3(M/M_\odot)$ km is the Schwarzschild radius of the massive object. In other words, the energy conversion efficiency is 50% for black holes ($r_* = r_s$) and $\sim 20\%$ for neutron stars ($r_* \sim 10$ km, $M \sim 1.4M_\odot$). In comparison, hydrogen nuclear fusion ($4{}^1_1\mathrm{H} \rightarrow {}^4_2\,\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma$) converts four protons into a helium with an energy conversion efficiency of $(4m_\mathrm{p} - m_\mathrm{He})/(4m_\mathrm{p}) \sim 0.007$, i.e., only less than 1% of the rest mass energy is released. Accretion onto compact objects is therefore an extremely powerful energy source.

1 Bondi Accretion

Consider a compact object embedded in a uniform medium of density ρ_∞ and sound speed $c_{s,\infty}$. The compact object is represented by a point mass M, and medium is an ideal fluid (no shocks) whose self-gravity is negligible compared to M. The system is in a steady state ($\partial/\partial t = 0$) such that the continuity and momentum equations become

$$\begin{aligned} \nabla \cdot (\rho \mathbf{v}) &= 0, && (1) \\ \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\nabla P}{\rho} + \nabla \Phi &= 0. && (2) \end{aligned}$$

Since the system is spherically symmetric, the contuinity equation can be written as

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}(r^2 \rho v) &= 0, \\ \Rightarrow 2r\rho v + r^2 \rho \frac{dv}{dr} + r^2 v \frac{d\rho}{dr} &= 0, \\ \Rightarrow \frac{2}{r} + \frac{1}{v}\frac{dv}{dr} + \frac{1}{\rho}\frac{d\rho}{dr} &= 0. && (3) \end{aligned}$$

Note that we have replaced $\partial/\partial r$ with d/dr because of the steady-state assumption. Similarly, the momentum equation in spherical coordinates is

$$v\frac{dv}{dr} + \frac{1}{\rho}\frac{dP}{dr} + \frac{GM}{r^2} = 0. \tag{4}$$

1.1 Isothermal gas

For isothermal gas $P = \rho c_s^2$, the momentum equation becomes

$$v\frac{dv}{dr} + \frac{c_s^2}{\rho}\frac{d\rho}{dr} + \frac{GM}{r^2} = 0. \tag{5}$$

Eliminating ρ using Eq. 3, we obtain

$$\begin{aligned} v\frac{dv}{dr} - c_s^2\left(\frac{2}{r} + \frac{1}{v}\frac{dv}{dr}\right) + \frac{GM}{r^2} &= 0, \\ \Rightarrow \frac{1}{2}\frac{dv^2}{dr}\left(1 - \frac{c_s^2}{v^2}\right) &= -\frac{GM}{r^2}\left(1 - \frac{r}{r_c}\right), \end{aligned} \tag{6}$$

where

$$r_c \equiv \frac{GM}{2c_s^2}. \tag{7}$$

Before we solve Eq. 6, we shall first examine its qualitative properties. The right-hand side contains two terms that scale with r^{-2} and r^{-1}, respectively. Therefore, we expect the right-hand side to be negative at small r (when the r^{-2} term dominates), crossing zero at r_c, and then turning positive at large r[1]. At $r = r_c$, the left-hand side also vanishes, which can be realized by $v^2 = c_s^2$, allowing two types of solutions:

(i) $dv^2/dr < 0$, and v^2 goes from supersonic at $r < r_c$ to subsonic at $r > r_c$.

(ii) $dv^2/dr > 0$, and v^2 goes from subsonic at $r < r_c$ to supersonic at $r > r_c$.

The first solution corresponds to the spherical accretion that we are studying, while the second solution corresponds to the "stellar wind" problem, where gas is ejected outward by a central star. Eq. 6 applies to both types of problems as v appears quadratically. In both cases, the gas is accelerated from subsonic to supersonic speeds, and the location where this transition happens (r_c) is called the *transonic point* (or just *sonic point*).

There are, in fact, another family of solutions to Eq. 6 that does not involve a transition from subsonic to supersonic flows. This is possible because the left-hand side can also vanish if $dv^2/dr = 0$. Therefore, we can have thw following types of solutions:

(iii) v^2 increases with r at small r, flattens out and reaches a maximum ($< c_s^2$) at r_c (i.e., $dv^2/dr|_{r=r_c} = 0$), and then decreases at larger r. The flow remains *subsonic* at all r.

(iv) v^2 decreases inversely with r at small r, reaches a minimum ($> c_s^2$) at r_c (i.e., $dv^2/dr|_{r=r_c} = 0$), and then increases at larger r. The flow remains *supersonic* at all r.

For spherical accretion, types (i) and (iii) are both plausible solutions. Type (iii) corresponds to the case where gas slowly settles to hydrostatic equilibrium. In contrast, type (i) represents

[1] It can also remain positive or negative at all r. However, that corresponds to double-valued solutions where there are two v's at a given r.

a more likely case where gas is accelerated to supersonic speeds at the sonic point, losing causal contact with the ambient medium and starting to free-fall onto the central object.

We are now in a good position to solve Eq. 6, which can be done by separation of variables:

$$\int \frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right) dv^2 = \int \frac{GM}{r^2}\left(\frac{r}{r_c}-1\right) dr$$
$$\Rightarrow \frac{v^2}{c_s^2} - \ln\frac{v^2}{c_s^2} - 4\frac{r_c}{r} - 4\ln\frac{r}{r_c} + C = 0 \tag{8}$$

where C is the integration constant. Different values of C corresponds to different curves in the r/r_c vs. v/c_s plane. The Bondi accretion and stellar wind solutions correspond to $C = 3$, while the solutions without a sonic point correspond to $C < 3$. The $C > 3$ solutions are unphysical as they have double-valued v's at a given r.

Since the system is in a steady state, the accretion rate $\dot{M} = 4\pi r^2 \rho v$ must be the same everywhere, or else there would be an accumulation of mass, violating the steady-state assumption. At large r, Eq. 8 becomes

$$\ln\frac{v^2}{c_s^2} + \ln\frac{r^4}{r_c^4} + C = 0 \Rightarrow vr^2 = e^{C/2} c_s r_c^2. \tag{9}$$

Since $\dot{M}$ at the sonic point is the same as $\dot{M}$ at large r: we obtain the accretion rate

$$\dot{M} = 4\pi r^2 \rho_\infty v = 4\pi e^{C/2} r_c^2 \rho_\infty c_s = 4\pi e^{C/2} \frac{G^2M^2}{4c_s^4} c_s \rho_\infty = \pi e^{3/2} \frac{G^2M^2}{c_s^3} \rho_\infty. \tag{10}$$

The Bondi accretion rate scales with M^2 and thus is sensitive to the mass of the central object.