Lecture 4 – Hydrostatic Equilibrium

A fluid is in hydrostatic equilibrium when the gravity acting on it (either external or from its own) is balanced by the pressure gradient force such that the fluid is at rest:

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} - \nabla\Phi = 0 \Rightarrow \nabla P = -\rho\nabla\Phi. \tag{1}$$

1 Spherically symmetric, self-gravitating gas

A spherically symmetric object held together under its own gravity ("self-gravity") is frequently encountered in astrophysics: stars, molecular clouds, planets (gas giants) are the classical examples. In addition, objects consist of stars (star clusters) or even dark matter (dark matter halos) are also governed by similar physics, where random motion of particles plays the same role of thermal pressure – countering the gravitational collapse.

In spherical coordinates, assuming spherically symmetric, the gradient of a scalar field is simply $\nabla f = \partial_r f$, so the condition of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\rho\frac{d\Phi}{dr} = -\rho g. \tag{2}$$

Since we consider self-gravity, the gravitational field

$$g(r) = \frac{GM(r)}{r^2} = \frac{G}{r^2}\int_0^r 4\pi r^2 \rho(r) dr \tag{3}$$

depends on the radial mass (or density) profile[1], which in turn depends on the radial profile of pressure.

[1]The "radial profile" of a quantity Q is the distribution of Q in the radial direction, i.e., Q as a function of radius.

1.1 Isothermal gas

1.1.1 Singular isothermal sphere

For isothermal gas, $P = \rho c_s^2$, where the sound speed $c_s = (\partial P/\partial \rho)^{1/2}$ is a constant. Therefore, Eq. 2 becomes

$$\begin{aligned} c_s^2 \frac{d\rho}{dr} &= -\rho \frac{G}{r^2} \int_0^r 4\pi r^2 \rho(r) dr, \\ \Rightarrow \frac{c_s^2}{4\pi G} r^2 \frac{1}{\rho}\frac{d\rho}{dr} &= -\int_0^r r^2 \rho(r) dr, \\ \Rightarrow \frac{c_s^2}{4\pi G} \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{1}{\rho}\frac{d\rho}{dr}\right) &= -\rho(r). \end{aligned} \tag{4}$$

If we assume density follows a power-law profile (an educated guess), $\rho(r) = Cr^{-\alpha}$, we can substitute it in the above ODE: $\rho^{-1} d\rho/dr = d(\ln \rho)/dr = -\alpha/r \Rightarrow -c_s^2\alpha/(4\pi G r^2) = Cr^{-\alpha}$, $\Rightarrow \alpha = 2 \Rightarrow C = c_s^2/(2\pi G)$. Therefore, we have found a solution

$$\rho(r) = \frac{c_s^2}{2\pi G r^2}. \tag{5}$$

This solution blows up ($\rho \to \infty$) at $r = 0$ and is known as the *singular isothermal sphere.*

1.1.2 Isothermal sphere with a core

We can avoid the unphysical boundary condition at $r = 0$ by requiring the density profile to flatten at the central region within a "core" with a characteristic radius r_c and density ρ_c. By doing so, we introduce a scale to the scale-free, self-similar power-law solution in Eq. 5.

Before we solve Eq. 4, let us re-examine the hydrostatic equilibrium condition in terms of the gravitational potential Φ:

$$\frac{d(\rho c_s^2)}{dr} = -\rho \frac{d\Phi(r)}{dr} \Rightarrow \int_{\rho_c}^{\rho} \frac{d\rho}{\rho} = -\frac{1}{c_s^2}\int_0^{\Phi} d\Phi \Rightarrow \rho(r) = \rho_c \exp\left[-\frac{\Phi(r)}{c_s^2}\right] \tag{6}$$

where we define $\Phi = 0$ at the center. Substituting $c_s^2 d(\ln \rho)/dr = -d\Phi/dr$ in Eq. 4, we recover the Poisson equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = \nabla^2 \Phi(r) = 4\pi G \rho(r). \tag{7}$$

in spherical coordinates.

It is often useful to cast equations into a dimensionless form. We can define the dimensionless gravitational potential $\psi = \Phi/c_s^2$ such that $\rho = \rho_c e^{-\psi} \Rightarrow d(\ln \rho)/dr = -d\psi/dr$, and Eq. 4, becomes

$$\frac{c_s^2}{4\pi G \rho_c}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = e^{-\psi} \tag{8}$$

We now can make radius dimensionless by introducing $\xi = r/r_c$ where $r_c = \sqrt{c_s^2/(4\pi G \rho_c)}$, and the ODE becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi}, \tag{9}$$

which is known as the *Emden-Chandrasehkar equation*. We impose the boundary conditions: $\psi = 0$ and $d\psi/d\xi = 0$ at $\xi = 0$ In terms of ρ and r, this corresponds to $\rho = \rho_c$ and $d\rho/dr = 0$ at $r = 0$. In other words, we require that the density flattens to a finite value at the center rather than increases indefinitely.

There is no analytic solution to Eq. 9, but there are asymptotic solutions at small and large ξ. As $\xi \to 0$, $e^{-\psi} \approx 1 \Rightarrow \xi^2\psi' \approx \xi^3/3 \Rightarrow \psi \approx \xi^2/6$. Therefore, the density profile near the center is

$$\rho(r) = \rho_c/e^{\psi} \approx \frac{\rho_c}{1 + r^2/(6r_c^2)}. \tag{10}$$

On ther other hand, as $\xi \to \infty$, the asymptotic solution is $\psi \approx \ln(\xi^2/2)$, so the density profile becomes

$$\rho(r) \approx \frac{2\rho_c}{\xi^2} = \frac{2\rho_c r_c^2}{r^2} = \frac{c_s^2}{2\pi G r^2}, \tag{11}$$

recovering the singular isothermal sphere.

1.1.3 The Bonnor-Ebert sphere

In reality, clouds do not exist in a vacuum. Instead, a cloud can be embedded in a uniform medium that provides an external pressure confining the cloud. Suppose the medium has a density of ρ_t and a pressure $P_{\rm ext} = \rho_t c_s^2$, we can integrate the total mass of an isothermal sphere up to a truncation radius r_t where $\rho(r_t) = \rho_t$:

$$\begin{aligned} M &= \int_0^{r_t} 4\pi r^2 \rho dr = \int_0^{r_t} 4\pi (r_c\xi)^2 \rho_c e^{-\psi} r_c d\xi = 4\pi r_c^3 \rho_c \int_0^{r_t} \frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) d\xi \\ &= 4\pi r_c^3 \rho_c \left(\xi^2\psi'\right)|_{\xi=\xi_t} = \frac{c_s^3}{\sqrt{2}\pi^{1/2}G^{3/2}\rho_c^{1/2}}\frac{\rho_t^{1/2}}{\rho_t^{1/2}}\left(\xi^2\psi'\right)|_{\xi=\xi_t} = \frac{c_s^4}{P_{\rm ext}^{1/2}G^{3/2}}m, \end{aligned} \tag{12}$$

where

$$m = \frac{1}{\sqrt{2}\pi^{1/2}}\frac{\rho_t^{1/2}}{\rho_c^{1/2}}\left(\xi^2\psi'\right)|_{\xi=\xi_t} \tag{13}$$

is a dimensionless number. Numerically, m has a maximum value of 1.18 at $\rho_c/\rho_t = 14$. We can define the *Bonnor-Ebert mass*:

$$M_{\rm BE} = 1.18\frac{c_s^4}{G^{3/2}P_{\rm ext}^{1/2}} = 1.18\frac{c_s^3}{G^{3/2}\rho_t^{1/2}} \tag{14}$$

which represents the maximum mass a cloud in hydrostatic equilibrium in the ambient density of ρ_t and sound speed of c_s. More massive clouds are unstable as hydrostatic equilibrium cannot be established and thus they would undergo gravitational collapse.

1.2 Polytropic gas

For a polytropic gas, $P = A\rho^\gamma$ where A is a constant. As opposed to the isothermal gas, the sound speed $c_s = (dP/d\rho)^{1/2} = (\gamma A\rho^{\gamma-1})^{1/2}$ is no longer a constant[2] but is now a function of density (and thus radius). Therefore, we should put c_s^2 inside the bracket of Eq. 4:

$$\begin{aligned} \frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2 c_s^2 \frac{1}{\rho}\frac{d\rho}{dr}\right) &= -\rho, \qquad (15)\\ \Rightarrow \frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \gamma A\rho^{\gamma-1} \frac{1}{\rho}\frac{d\rho}{dr}\right) &= -\rho. \qquad (16)\end{aligned}$$

Introducing the polytropic index $n = (\gamma-1)^{-1} \Rightarrow \gamma = 1/n + 1 \Rightarrow \gamma - 2 = 1/n - 1$. Let $\rho = \rho_c\theta^n \Rightarrow d\rho/dr = n\rho_0\theta^{n-1}d\theta/dr \Rightarrow \rho^{\gamma-2} = (\rho_c\theta^n)^{1/n-1} = \rho_c^{1/n-1}\theta^{1-n}$. Substituting in Eq. 15,

$$\begin{aligned} \frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\frac{1}{n}+1\right)A\rho_c^{1/n-1}\theta^{1-n}n\rho_0\theta^{n-1}\frac{d\theta}{dr}\right] &= -\rho_c\theta^n,\\ \Rightarrow \frac{A(1+n)\rho_c^{1/n-1}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) &= -\rho_c\theta^n. \qquad (17)\end{aligned}$$

Define the core radius

$$r_c = \left[\frac{A(1+n)\rho_c^{1/n-1}}{4\pi G}\right]^{1/2} = \left[\frac{A(1+n)P_c}{4\pi G\rho_c^2}\right]^{1/2}, \qquad (18)$$

where $P_c = A\rho_c^{1/n+1}$, we can make the radius dimensionless by letting $\xi = r/r_c$. The ODE now becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \qquad (19)$$

This is known as the *Lane-Emden equation*, with the boundary conditions: $\theta = 1$ and $\theta' = 0$ at $\xi = 0$. Analytic solutions only exist for $n = 0, 1, 5$, while other cases have to be integrated numerically. Isothermal gas corresponds to $n \to \infty$, which needs to be treated separately as we did in the previous section. For $n < 5$, θ drops to zero at a finite radius, and so we can define the size of the sphere when $\theta(\xi_{\max}) = 0$.

1.2.1 The mass-size relation of a polytropic sphere

A polytropic sphere has a size of $R = \xi_{\max}r_c \propto \rho_c^{(1-n)/2n}$ and a mass of

$$M = \int_0^R 4\pi r^2\rho dr = \int_0^{\xi_{\max}} 4\pi r_c^2\xi^2\rho_c\theta^n r_c d\xi = 4\pi r_c^3\rho_c(\xi^2|\theta'|)|_{\xi=\xi_{\max}} \propto \rho_c^{(3-n)/(2n)}. \qquad (20)$$

[2]Note that the isothermal gas is a special case where $\gamma = 1$.

The mass-size relation is therefore

$$M \propto \rho_c^{(3-n)/(2n)} \propto R^{\frac{2n}{1-n}\frac{3-n}{2n}} \propto R^{(3-n)/(1-n)} \tag{21}$$

A white dwarf is a stellar remnant where the electron degenerate pressure dominates over thermal pressure and can be well-described by non-relativistic degenerate gas, $n = 1.5$ ($\gamma = 5/3$), Therefore, its mass-size relation is

$$M \propto R^{\frac{3-1.5}{1-1.5}} \propto R^{-3}. \tag{22}$$

White dwarfs therefore have the counter-intuitive property that its size shrinks as it accretes more mass. As a white dwarf becomes more massive, its particle velocities become comparable to the speed of light, and we need to consider the relativistic effect. For relativistic degenerate gas, $n = 3$ ($\gamma = 4/3$), the mass-size relation is

$$M \propto R^{\frac{3-3}{1-3}} \propto R^0, \tag{23}$$

and thus M is independent of R. This implies that there is a maximum mass of a white dwarf $M \approx 1.4 M_\odot$, above which hydrostatic equilibrium can no longer be established. This is known as the *Chandrasehkar mass*.