Lecture 2 – Dissipation

1 Viscosity

In the previous lecture, we have assumed that there is no dissipation in the fluid (ideal fluid). In reality, the relative motion between fluid elements causes dissipative forces called *viscosity*. A familar example of a viscous fluid is honey (high viscosity), which is more difficult to stir compared to water (low viscosity). The stickiness of honey comes from viscosity.

1.1 Velocity Gradient

The relative motion between fluid elements can be described by the *velocity gradient* tensor

$$\nabla \mathbf{v} = \partial_i v_j. \tag{1}$$

We can decompose $\nabla \mathbf{v}$ into three *irreducible* tensors:

$$\begin{aligned} \nabla \mathbf{v} = \partial_i v_j &= \frac{1}{2}(\partial_i v_j + \partial_j v_i) + \frac{1}{2}(\partial_i v_j - \partial_j v_i) \\ &= \frac{1}{3}\partial_k v_k \delta_{ij} + \frac{1}{2}(\partial_i v_j + \partial_j v_i) - \frac{1}{3}\partial_k v_k \delta_{ij} + \frac{1}{2}(\partial_i v_j - \partial_j v_i) \\ &= \frac{1}{3}\theta \mathbb{I} + \mathbb{S} + \mathbb{R}, \end{aligned} \tag{2}$$

where θ is the *expansion rate* (scalar), $\mathbb{S}$ is the trace-free, symmetric *shear tensor*, and $\mathbb{R}$ is the anti-symmetric *rotation tensor*, respectively defined as

$$\theta = \nabla \cdot \mathbf{v}, \tag{3}$$

$$\mathbb{S} = \frac{1}{2}\left(\nabla \mathbf{v} + \nabla \mathbf{v}^{\intercal} - \frac{1}{3}\theta \mathbb{I}\right) \tag{4}$$

$$\mathbb{R} = \frac{1}{2}\left(\nabla \mathbf{v} - \nabla \mathbf{v}^{\intercal}\right). \tag{5}$$

The prefactor 1/3 is such that the shear tensor is trace-free: $\mathrm{Tr}(\mathbb{S}) = S_{ii} = \partial_i v_i - \partial_k v_k \delta_{ii}/3 = 0$. Note that $\delta_{ii} = 3$ (the trace of an identity matrix) and that i and k are dummy variables. The veolcity divergence changes the volume of the fluid via isotropic expansion/compression, keeping the shape of the fluid element fixed. On the other hand, the shear tensor changes a fluid element's shape while keeping its volume fixed (as it's trace-less). The rotation tensor controls, well, the rotation of the fluid element, which can be seen by noting that it is a linear combination of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$:

$$\frac{1}{2}\epsilon_{ijk}\omega_k = \frac{1}{2}\epsilon_{ijk}\epsilon_{abk}\partial_a v_b = \frac{1}{2}\left(\partial_i v_j - \partial_j v_i\right) = R_{ij}. \tag{6}$$

The velocity gradient is a rank 2 tensor in 3D and thus has $3 \times 3 = 9$ degrees of freedom (d.o.f.). On the other hand, the velocity divergence is a scalar (d.o.f. = 1); the rotation tensor is antisymmetric (d.o.f. = 3); the shear tensor is a symmetric tensor (d.o.f. = 6), but it is also trace-free, which is an additional constrant, so its d.o.f. $= 6 - 1 = 5$. In other words, the d.o.f. of the three tensors add up to $1 + 3 + 5 = 9$, as expected.

1.2 Momentum Equation with Viscosity (Navier-Stokes Equation)

We assume *Newtonian fluids*, where the viscous stress tensor is *linearly proportional* to the velocity gradient (Hooke's law: stress $\propto$ strain). In this case, the *viscous stress tensor* can be written as

$$\mathbb{T}_{\rm visc} = -(\mu\theta\mathbb{I} + 2\eta\mathbb{S}), \tag{7}$$

where μ and η are the coefficients of the bulk viscosity and shear viscosity, respectively. There is no corresponding term for the rotation tensor, as rotation does not lead to a change in volume or shape. We can now add an extra force term to our momentum equation:

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P - \nabla\mathbb{T}_{\rm visc} = -\nabla P + \nabla(\mu\theta) + 2\nabla\cdot(\eta\mathbb{S}) \tag{8}$$

This is the famous *Navier-Stokes equation.* If we further assume *incompressible* flows ($\theta = \nabla\cdot\mathbf{v} = 0$), the bulk viscosity term vanishes, while the shear viscosity term can be expanded into $2\eta\nabla\cdot\mathbb{S} = \eta\partial_i(\partial_i v_j + \partial_j v_i) = \eta[\nabla^2\mathbf{v} + \nabla(\nabla\cdot\mathbf{v})] = \eta\nabla^2\mathbf{v}$, assuming η is spatially constant. We therefore obtain the Navier-Stokes equation in its most common form:

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} + \nu\nabla^2\mathbf{v}, \tag{9}$$

where $\nu = \eta/\rho$ is the *kinetic viscosity.* Viscosity results in the diffusion of the velocity field, which smoothes out shear flows.

1.3 Energy Equation with Viscous Dissipation

Just like pressure provides an energy flux of $P\mathbf{v}$, viscosity also provides an energy flux of $\mathbb{T}_{\rm visc}\mathbf{v}$. Therefore, we have an extra term in the energy equation:

$$\frac{\partial(\rho e)}{\partial t} + \nabla\cdot[(\rho e + P)\mathbf{v}] + \nabla\cdot(\mathbb{T}_{\rm visc}\mathbf{v}) = 0. \tag{10}$$

Using the continuity equation, this reduces to

$$\rho\frac{de}{dt} + \nabla\cdot(P\mathbf{v}) + \nabla\cdot(\mathbb{T}_{\rm visc}\mathbf{v}) = 0. \tag{11}$$

Expanding each term: $de/dt = \mathbf{v} \cdot d\mathbf{v}/dt + du/dt$, $\nabla \cdot (P\mathbf{v}) = \mathbf{v} \cdot \nabla P + P\nabla \cdot \mathbf{v}$, and $\nabla \cdot (\mathbb{T}_{\text{visc}}\mathbf{v}) = \partial_i(T_{ij}v_j) = v_j\partial_i T_{ij} + T_{ij}\partial_i v_j = \mathbf{v} \cdot (\nabla \cdot \mathbb{T}_{\text{visc}}) + \mathbb{T}_{\text{visc}} : \nabla\mathbf{v}$. We can see that there is a $\mathbf{v} \cdot (...)$ in each term, and that they add up to the Navier-Stokes equation (Eq. 8). Dropping there terms, the energy equation reduces to

$$\rho \frac{du}{dt} = -P\nabla \cdot \mathbf{v} - \mathbb{T}_{\text{visc}} : \nabla\mathbf{v}. \tag{12}$$

Note that $\mathbb{A} : \mathbb{B} \equiv A_{ij}B_{ij} = A_{ij}B^{\intercal}_{ji} = \text{Tr}(\mathbb{A}\mathbb{B}^{\intercal})$. Therefore, in addition to adiabatic contraction, the fluid can also gain energy via viscous dissipation, which transform the internal relative motion (i.e., velocity gradient) into heat.

It is perhaps more informative to express the energy equation in terms of entropy. Using the first law of thermodynamics and the continuity equation, we obtain

$$\rho T \frac{ds}{dt} = -\mathbb{T}_{\text{visc}} : \nabla\mathbf{v} = (\mu\theta\mathbb{I} + 2\eta\mathbb{S}) : \left(\frac{1}{3}\theta\mathbb{I} + \mathbb{S} + \mathbb{R}\right). \tag{13}$$

Contracting a symmetric tensor with an anti-symmetric tensor leads to zero and thus $\mathbb{R}$ drops out. Also, the term $\mathbb{S} : \mathbb{I}$ and $\mathbb{I} : \mathbb{S}$ vanish as $\mathbb{S}$ is trace-free. Therefore, we are left with

$$\rho T \frac{ds}{dt} = \mu\theta^2 + 2\eta\mathbb{S} : \mathbb{S} \geq 0, \tag{14}$$

as $\mathbb{S} : \mathbb{S} = S_{ij}S_{ij}$ is just the quadratic sum of all the 9 components of $\mathbb{S}$. Entropy is no longer conserved in viscous fluids. It can be generated via dissipation and convert kinetic energy into heat, but it can never drop, in line with the second law of thermodynamics.