

Lecture 1 – Ideal Fluid

1 Fluid Description

A fluid is continuous medium that can deform easily and has well-defined macroscopic properties such as density, velocity, and pressure. Fluid mechanics can be applied to gas and liquids, and it shares a lot of similarities to plasma physics as well as continuum mechanics (which describes solids).

In the fluid description, the minimum component is the *fluid element*, within which the macroscopic thermodynamical properties (e.g., density, pressure, temperature) are uniform. In order for the fluid description to hold, the typical length scale of interest $L = Q/\nabla Q$ must be significantly larger than the microscopic particle mean free path λ_{mfp} . We can define the Knudsen number as $\text{Kn} \equiv \lambda_{\text{mfp}}/L$. The fluid description is valid when $\text{Kn} \ll 1$. Otherwise ($\text{Kn} \gtrsim 1$), we need to resort to kinetic theory, which is the basis of plasma physics.

2 Conservation Laws for Ideal Fluids

In this lecture, we will focus on the *ideal fluids* (or *perfect fluids*) where dissipation such as viscosity and thermal conduction are negligible (dissipation will be discussed in the next lecture). Ideal fluids do not exist in nature, but they offer a theoretical foundation to which we can add more complex and realistic physics. They can even be a very good approximation in many situations, especially in astrophysics.

The equations of fluid dynamics consist of the conservations laws of three conserved quantities: density, momentum, and energy.

2.1 Conservation of Mass

The equation for mass conservation, often referred to as the “continuity equation”, takes the following form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

where ρ and \mathbf{v} are the density and velocity of the fluid, respectively. This is the *differential* form of the continuity equation. We can integrate the equation over a small volume element,

$$\frac{d}{dt} \int \rho dV + \int \nabla \cdot (\rho \mathbf{v}) dV = \frac{dM}{dt} + \oint \rho \mathbf{v} \cdot d\mathbf{A} = 0, \quad (2)$$

which is the *integral* form of the continuity equation. The physical meaning is clear: the rate of change of the mass in the volume is controlled by the mass flowing in and out of the surface.

Note that Eq. 1 is in analogy to the charge conservation equation, $\partial\rho_e/\partial t + \nabla \cdot \mathbf{j} = 0$, in electromagnetism. This is no coincidence. In general, for any conserved quantity, there is a corresponding equation in the form of

$$\frac{\partial(\text{density})}{\partial t} + \nabla \cdot (\text{flux}) = 0. \quad (3)$$

Eq. 1 is what we call the *Eulerian* description, where we describe the evolution of fluid quantities at a fixed location. There is another, equally important description that we shall introduce here. By expanding the second term in Eq. 1, $\nabla \cdot (\rho\mathbf{v}) = \partial_i(\rho v_i) = v_i\partial_i\rho + \rho\partial_i v_i = \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v}$, we can rewrite the continuity equation as

$$\frac{\partial\rho}{\partial t} + \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v} = \frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{v} = 0, \quad (4)$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (5)$$

is the *material derivative* (or *convective derivative*) which describes the rate of change of a fluid quantity as we *come with the fluid!* This is called the *Lagrangian* description. Rearranging terms, we get

$$\frac{d\rho}{dt} = -\rho\nabla \cdot \mathbf{v}, \quad (6)$$

which is the Lagrangian form of the continuity equation.

For a fluid element of volume V and mass $M = \rho V$, we have $dM/dt = V d\rho/dt + \rho dV/dt = 0$, and thus

$$\frac{1}{\rho} \frac{d\rho}{dt} = -\frac{1}{V} \frac{dV}{dt} = -\nabla \cdot \mathbf{v}. \quad (7)$$

We can define $\theta = \nabla \cdot \mathbf{v}$ as the *rate of expansion*, describing the fractional change of the volume of a fluid element.

2.2 Conservation of Momentum

To write down the equation of momentum conservation (or, the “equation of motion”), we first identify the momentum density as $\rho\mathbf{v}$ and the momentum flux as $\rho\mathbf{v} \otimes \mathbf{v}$. It is tempting to write $\partial(\rho\mathbf{v})/\partial t + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) = 0$ following the logic from the previous section. However, if we expand the terms $\partial(\rho\mathbf{v})/\partial t = \mathbf{v}\partial\rho/\partial t + \rho\partial\mathbf{v}/\partial t$ and $\nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) = \partial_i(\rho v_i v_j) = v_j\partial_i(\rho v_i) + \rho v_i\partial_i v_j = \mathbf{v}\nabla \cdot (\rho\mathbf{v}) + \rho\mathbf{v} \cdot \nabla\mathbf{v}$, it follows that

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) = \rho\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} = \rho\frac{d\mathbf{v}}{dt}, \quad (8)$$

where we have used the continuity equation (Eq. 1) to drop terms. From Newton's second law ($F = ma$), we know there has to be a corresponding force (per volume) term \mathbf{f} . In the simplest case (ignoring dissipation, gravity, etc.), the force is sourced by pressure (or pressure gradients):

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho}, \quad (9)$$

where the minus sign accounts for the fact that the pressure force goes from high-pressure regions to low-pressure regions. Eq. 9 is the Lagrangian form of the momentum equation. If we want to write a corresponding Eulerian form, we can put the pressure term to the left hand side:

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v} + P\mathbb{I}) = 0, \quad (10)$$

where \mathbb{I} is the identity tensor and is here just so we can put the pressure term into a divergence. Eq. 10 is the Eulerian form of the momentum equation.

2.3 Conservation of Energy

In an ideal fluid, its thermal energy can be transformed into kinetic energy and vice versa, but the total energy (i.e., kinetic + thermal) is conserved. For a fluid element of volume V and mass M , its total energy is $E = M|\mathbf{v}|^2/2 + U$ where U is the thermal energy. In the Lagrangian description, we often adopt the “specific energy”, which means energy *per mass*,

$$e = \frac{1}{2}|\mathbf{v}|^2 + u, \quad (11)$$

where $e = E/M$ and $u = U/M$. Multiplying by ρ , we obtain the “energy density” (energy *per volume*)

$$\rho e = \frac{1}{2}\rho|\mathbf{v}|^2 + \rho u, \quad (12)$$

which is used in Eulerian description.

At this point, it might be tempting to write down the energy conservation equation as $\partial(\rho e)/\partial t + \nabla \cdot (\rho e\mathbf{v}) = 0$. However, this only describes the energy carried by the fluid flows and is not the whole story. As the fluid expands, its pressure force would also do work and thus transfer energy to its surrounding (per work-energy theorem). As such, there should be an extra term in the divergence accounting for that, and the complete energy equation reads

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot [(\rho e + P)\mathbf{v}] = 0. \quad (13)$$

We shall now derive the Lagrangian form of the energy equation. Similar to our derivation of the momentum equation, $\partial(\rho e)/\partial t + \nabla \cdot (\rho e\mathbf{v})$ can be reduced to $\rho(de/dt)$ with the help of

the continuity equation, and we are left with

$$\rho \frac{de}{dt} + \nabla \cdot (P\mathbf{v}) = 0. \quad (14)$$

which is already in Lagrangian form. However, it is often desirable to treat u as the primary variable instead of e . We can expand

$$\frac{de}{dt} = \frac{d}{dt} \left(\frac{1}{2} |\mathbf{v}|^2 + u \right) = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{du}{dt}, \quad (15)$$

and plug it into 14 and obtain

$$\rho \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{du}{dt} \right) + \mathbf{v} \cdot \nabla P + P \nabla \cdot \mathbf{v} = 0. \quad (16)$$

Since the evolution of velocity is already governed by the momentum equation, we can take its Lagrangian form, $d\mathbf{v}/dt = -\nabla P/\rho$, and plug it into Eq. 16, and get

$$\frac{du}{dt} = -\frac{P}{\rho} \nabla \cdot \mathbf{v}, \quad (17)$$

which is the Lagrangian form of the (thermal) energy equation we were after. The physical meaning is clear: for an adiabatic, dissipationless fluid, the only way to vary its thermal energy is by expansion/contraction.

We can gain further insights using the first law of thermodynamics, $dU = TdS - PdV$, where T is the temperature and S is the entropy. Express it in the specific form (i.e., per mass):

$$du = Tds - \frac{P}{M} dV = Tds + \frac{P}{\rho^2} d\rho, \quad (18)$$

where we have used the relation $dV/V = -d\rho/\rho$ in the second step. It follows that

$$T \frac{ds}{dt} = \frac{du}{dt} - \frac{P}{\rho^2} \frac{d\rho}{dt} = -\frac{P}{\rho} \left(\nabla \cdot \mathbf{v} + \frac{1}{\rho} \frac{d\rho}{dt} \right) = 0, \quad (19)$$

where we have used Eq. 17 and Eq. 6 in the second and third steps, respectively. The specific entropy is conserved in an adiabatic, dissipationless fluid.

3 Equation of State

We have seen that the conservation laws are described by 5 equations: 1 for density, 3 for velocity (in x, y, z directions), and 1 for energy. However, there are 6 unknown variables: $(\rho, v_x, v_y, v_z, P, u)$. In order to solve (or to “close”) the system of equations, we need another equation. The equation of state describes the relation between pressure and thermal energy. Assuming we have an *ideal gas* (not to be confused with *ideal fluid*), the ideal gas law states

that

$$P = \frac{\rho}{\mu m_p} k_B T, \quad (20)$$

where m_p is the proton mass, k_B is the Boltzmann constant, and μ is the mean molecular weight (depends on the composition of gas: $\mu = 1$ for monatomic gas; $\mu = 2$ for diatomic molecular gas). Meanwhile, the thermal energy is related to temperature via

$$u = \frac{N k_B T}{2 \mu m_p} = \frac{1}{\gamma - 1} \frac{k_B T}{\mu m_p} \quad (21)$$

where N is the degrees of freedom. For monatomic gas, $N = 3$. For diatomic gas, N can be 3, 5, or 7 depending on whether the rotational and vibrational degrees of freedom are excited. Each degree of freedom contributes $k_B T/2$ to the thermal energy (the equipartition principle). Alternatively, we can use the *adiabatic index* $\gamma = 1 + 2/N$. Combining Eqs. 20 and 21, we obtain the equation of state

$$P = (\gamma - 1)\rho u, \quad (22)$$

which closes the system.

The need of an extra equation to close the system of equations is a very general problem and can be found in many places such as kinetic theory, plasma physics, stellar dynamics, or radiation transfer. One must adopt certain approximations (like our equation of state) in order to close the system. This is known as the “closure problem”.

4 Incompressible Flows

A fluid is said to be incompressible if the velocity divergence vanishes:

$$\nabla \cdot \mathbf{v} = 0. \quad (23)$$

A direct consequence is that the density of the fluid remain constant as $d\rho/dt = -\rho\nabla \cdot \mathbf{v} = 0$. Compressibility is actually a *flow property*: it depends on how the fluid velocity compares to the speed of sound. If the fluid is moving much slower than the speed of sound, any small compression would propagate away as sound waves almost instantaneously, and the flow becomes incompressible.

In incompressible flows, Eq. 23 provides an extra constraint. As such, the energy equation is decoupled from the continuity equation and momentum equation as the latter two are closed by the extra equation, Eq. 23, and so there are 5 unknowns and 5 equations. If desired, energy can be computed separately using the energy equation, otherwise it can be neglected completely.

Note that pressure only exists in our incompressible fluid equations as a gradient. Therefore, the absolute value of pressure does not matter, and we can add an arbitrary constant to the pressure without changing the equation of motion. In other words, pressure can only be

determined up to a constant, like a potential. If we take the divergence of the momentum equation, we obtain

$$\frac{\partial(\nabla \cdot \mathbf{v})}{\partial t} + \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla \cdot \left(\frac{\nabla P}{\rho} \right). \quad (24)$$

Assuming ρ is constant everywhere, the momentum equation becomes

$$\nabla^2 P = -\rho \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}). \quad (25)$$

This is a Poisson equation and can be solved using potential theory. Pressure in incompressible flows plays a special role: it readjusts itself instantaneously such that the velocity divergence remains zero at all times.